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# Gravity waves, scale asymptotics and the pseudo-incompressible equations

ULRICH ACHATZ[1]†, R. KLEIN[2] AND F. SENF[3]

[1]Institut für Atmosphäre und Umwelt, Goethe-Universität Frankfurt, Altenhöferallee 1, 60438 Frankfurt am Main, Germany

[2]Institut für Mathematik, Freie Universität Berlin, Arnimallee 6, 14195 Berlin-Dahlem, Germany

[3]Leibniz-Institut für Atmosphärenphysik an der Universität Rostock, Schlossstraße 6, 18225 Kühlungsborn, Germany

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Multiple-scale asymptotics is used to analyse the Euler equations for the dynamical situation of a gravity wave (GW) near breaking level. A simple saturation argument in combination with linear theory is used to obtain the relevant dynamical scales. As a small expansion parameter, the ratio of the inverse of the vertical wavenumber and potential temperature and pressure scale heights is used, which we allow to be of the same order of magnitude here. It is shown that the resulting equation hierarchy is consistent with that obtained from the pseudo-incompressible equations, both for non-hydrostatic and hydrostatic GWs, while this is not the case for the anelastic equations unless the additional assumption of sufficiently weak stratification is adopted. To describe vertical propagation of wavepackets over several atmospheric-scale heights, Wentzel–Kramers–Brillouin (WKB) theory is used to show that the pseudo-incompressible flow divergence generates the same amplitude equation that also obtains from the full Euler equations. This gives a mathematical justification for the use of the pseudo-incompressible equations in the study of GW breaking in the atmosphere for arbitrary background stratification. The WKB theory interestingly even holds at wave amplitudes close to static instability. In the mean-flow equations, we obtain in addition to the classic wave-induced momentum-flux divergences a wave-induced correction of hydrostatic balance in the vertical momentum equation, which cannot be obtained from Boussinesq or anelastic dynamics.

**Key words:** atmospheric flows, internal waves, stratified flows

## 1. Introduction

The filtering of fast insignificant motion from the equations of atmospheric dynamics has a long history. It is useful in at least two respects: (i) it provides simplified equation systems which can help in gaining a deeper conceptual understanding of intricate processes, and (ii) it filters fast motions and thus yields dynamical descriptions which allow much longer time steps than the compressible Euler equations in numerical integrations.

A typical field for the application of filtered equation systems is the dynamics of gravity waves (GWs) (Fritts & Alexander 2003). Under the assumption that sound

† Email address for correspondence: achatz@iau.uni-frankfurt.de

waves only act to very rapidly adjust to a balanced state, so-called sound-proof equations are most often used in the study of GW dynamics. The dynamics is taken to be balanced with respect to acoustic modes at all times. For processes which have scales exclusively below the atmospheric scale heights, the Boussinesq equations provide an appropriate simplification. The filtering is achieved by requiring a non-divergent wind. Probably the major part of our present understanding of GW dynamics has been obtained from these equations. Nonetheless, for the description of important aspects of GW dynamics, we need more general equations. GWs are typically radiated upwards from the troposphere, and often cover large altitude differences before they interact with the large-scale flow (e.g. Lindzen 1981). An important aspect is the amplitude growth they experience in their propagation through an increasingly rarified medium. Wave growth finally leads to instabilities and turbulent breaking which then cause large-scale flow acceleration or deceleration and heating or cooling. This growth, however, cannot be obtained from the Boussinesq equations. A generalization is needed for which several examples exist.

Batchelor (1953) and Ogura & Phillips (1962) have developed what has come to be known as the anelastic equations. They require the wind weighted by the altitude-dependent mean background density of a hydrostatic reference atmosphere to be non-divergent. This density weighting induces the observed wave growth. The original anelastic equations, however, suffer from the basic assumption that the potential temperature of the reference atmosphere may only have a very weak vertical dependence, so that the leading-order anelastic divergence constraint involves an adiabatic background stratification. This is in stark contrast with realistic stratifications, where between the GW sources and the wave-breaking altitude potential temperature typically increases by more than one order of magnitude.

A generalization of the derivation of the anelastic equations has been given by Lipps & Hemler (1982) and Lipps (1990). Constant reference potential temperature is no longer required but it is still assumed that its vertical dependence is weak, and that the deviations of potential temperature from that of the reference atmosphere are small. An alternative approach is given by the pseudo-incompressible equations (Durran 1989; Durran & Arakawa 2007; Durran 2008). Here the argument is the explicit filtering of any dynamics which allows us an exchange between the elastic part of potential energy, which is carried by the pressure fluctuations, and kinetic energy. Recently, we have studied these approximations from a mathematical perspective (Klein 2000, 2009; Klein *et al.* 2010). For realistic stratifications stronger than those assumed by Ogura & Phillips (1962), a three time scale regime emerges, with sound propagation being fastest, internal waves of intermediate time scale, and advection the slowest. Thus, even sound-proof models in which the sound-wave propagation time scale is eliminated still involve fast internal wave motions and slow advection, i.e. they describe a two time-scale regime. Given this situation, a thorough multiple-scale analysis for the regime most important to internal-wave breaking seems in order.

The approach taken here is to consider GWs at the threshold of static instability, to characterize this flow regime by appropriate non-dimensional parameters and then to pursue systematic multiple-scale asymptotics. We show that the resulting equation system is, under rather general conditions, consistent with the pseudo-incompressible equations, which thus offer themselves as the most appropriate reduced sound-proof system for the study of GW dynamics near the breaking level. Specifically, in this analysis we allow for arbitrary background stratification and consider GWs for which the inverse of a typical vertical wavenumber is small compared to the pressure and potential-temperature scale heights.

This paper is structured as follows. In § 2, we review the results of linear GW theory needed here. These are combined in § 3 with a simple saturation argument to yield the dynamically meaningful scales of the problem. As a small parameter we introduce the ratio between the inverse of the vertical wavenumber and potential-temperature scale height, and carry out a multiple-scale asymptotic expansion for the Euler equations in this regime. In § 4 it is shown that a corresponding multiple-scale asymptotics of the pseudo-incompressible equations yields the same equation hierarchy, while this is not the case for the anelastic equations unless sufficiently weak stratification is assumed, so that the potential-temperature scale height is considerably larger than that of the Exner pressure. Note, however, that in the stratosphere, for example, stratification is comparable to isothermal or even stronger, while even in the mesosphere, where temperature actually decreases with height, potential temperature nonetheless has a significantly positive vertical gradient, so that a weak stratification is not an appropriate assumption there. The consistency between the compressible and pseudo-incompressible models is also shown for hydrostatic GWs in § 5. We move on to analyse the dynamics of GW packets, propagating over several atmospheric-scale heights, at small wave amplitudes in § 6 and large amplitudes in § 7. We conclude with a summary in § 8.

## 2. Linear gravity waves in an isothermal hydrostatic atmosphere

Consider the most simple example of GWs growing in their upward propagation due to the ambient density gradient: we neglect rotation, use only one dimension in the horizontal, and also focus on a local tangent plane in Cartesian coordinates. Then the inviscid Euler equations without heat sources can be written as

$$\frac{\mathrm{D}u}{\mathrm{D}t} + c_p\theta\frac{\partial\pi}{\partial x} = 0, \tag{2.1}$$

$$\frac{\mathrm{D}w}{\mathrm{D}t} + c_p\theta\frac{\partial\pi}{\partial z} = -g, \tag{2.2}$$

$$\frac{\mathrm{D}\theta}{\mathrm{D}t} = 0, \tag{2.3}$$

$$\frac{\mathrm{D}\pi}{\mathrm{D}t} + \frac{R}{c_v}\pi\nabla\cdot\boldsymbol{v} = 0. \tag{2.4}$$

Here $\boldsymbol{v} = (u, w)$ is the wind vector with horizontal and vertical components $u$ and $w$, respectively,

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + w\frac{\partial}{\partial z} \tag{2.5}$$

is the material derivative; $c_p$ and $c_v$ are the specific heat coefficients at constant pressure and volume, respectively, while $R = c_p - c_v$ is the gas constant; $\pi = (p/p_{00})^{R/c_p}$ is the Exner pressure, to be calculated from pressure $p$ and a reference value $p_{00}$, characterizing conditions at some reference altitude $z_{00}$. If $T$ is the temperature then $\theta = T/\pi$ is the potential temperature. Finally, $g$ is the gravitational acceleration. For later reference, we also note that density, $\rho$, potential temperature and Exner pressure are linked via the equation of state

$$R\rho\theta = p_{00}\pi^{(1-\kappa)/\kappa}, \tag{2.6}$$

where $\kappa = R/c_p$.

Consider now low-amplitude GWs in an atmosphere at rest. The latter, denoted by a bar, can only depend on altitude, and it must be in hydrostatic equilibrium, i.e.

$$c_p\overline{\theta}\frac{\partial\overline{\pi}}{\partial z} = -g. \tag{2.7}$$

We assume

$$\boldsymbol{v} = \boldsymbol{v}', \tag{2.8}$$

$$\theta = \overline{\theta} + \theta', \tag{2.9}$$

$$\pi = \pi + \pi', \tag{2.10}$$

where all perturbation quantities, denoted by primes, are infinitesimally small. Then the Euler equations become, under neglect of all nonlinear terms in the perturbation fields and using (2.7),

$$\frac{\partial u'}{\partial t} + c_p\overline{\theta}\frac{\partial \pi'}{\partial x} = 0, \tag{2.11}$$

$$\frac{\partial w'}{\partial t} + c_p\overline{\theta}\frac{\partial \pi'}{\partial z} = b', \tag{2.12}$$

$$\frac{\partial b'}{\partial t} + N^2 w' = 0, \tag{2.13}$$

$$c_p\overline{\theta}\frac{\partial \pi'}{\partial t} - g w' + \frac{R}{c_v}c_p\overline{\theta}\overline{\pi}\nabla\boldsymbol{\cdot}\boldsymbol{v}' = 0, \tag{2.14}$$

where $b' = g\theta'/\overline{\theta}$ is the perturbation buoyancy and $N^2 = (g/\overline{\theta})(\partial\overline{\theta}/\partial z)$ is the squared Brunt–Väisälä frequency. It is interesting to note that these equations conserve, via

$$\frac{\partial E'}{\partial t} + \frac{\partial p'u'}{\partial x} + \frac{\partial p'w'}{\partial z} = 0, \tag{2.15}$$

the perturbation energy

$$E' = \frac{\overline{\rho}}{2}\left(u'^2 + w'^2 + \frac{b'^2}{N^2} + \frac{c_p^2}{c_s^2}\overline{\theta}^2\pi'^2\right). \tag{2.16}$$

Here

$$\overline{\rho} = \frac{p_{00}}{R\overline{\theta}}\overline{\pi}^{(1-\kappa)/\kappa} \tag{2.17}$$

is the density of the reference atmosphere and $c_s = \sqrt{\gamma R\overline{T}}$ is the velocity of sound, with $\gamma = c_p/c_v$; $p' = c_p\overline{\rho}\overline{\theta}\pi'$ is the perturbation pressure. Clearly, as the ambient density decreases the wind amplitudes must increase.

In the special case of an isothermal atmosphere with $\overline{T} = T_{00} = \text{constant}$, $c_s$ is also a constant. From (2.7) and $\overline{\theta} = \overline{T}/\overline{\pi}$, one then gets $\overline{\pi} = \exp[-(z - z_{00})/H_\theta]$ and $\overline{\theta} = T_{00}\exp[(z - z_{00})/H_\theta]$, where $H_\theta = c_p T_{00}/g$ is the potential-temperature scale height. One also has $N^2 = g/H_\theta$. Likewise (2.17) yields $\overline{\rho} = \rho_{00}\exp[-(z - z_{00})/H]$. Here $H = RT_{00}/g$ is the density and pressure height, and $\rho_{00} = p_{00}/RT_{00}$. Thus motivated, we introduce rescaled fields $\boldsymbol{v}''$, $\pi''$ and $b''$ so that

$$\boldsymbol{v}' = \boldsymbol{v}''\exp\left(\frac{z - z_{00}}{2H}\right), \tag{2.18}$$

$$\pi' = \pi''\exp\left(\frac{z - z_{00}}{2H} - \frac{z - z_{00}}{H_\theta}\right), \tag{2.19}$$

$$b' = b'' \exp\left(\frac{z - z_{00}}{2H}\right). \tag{2.20}$$

The linearized Euler equations then become

$$\frac{\partial u''}{\partial t} + c_p T_{00} \frac{\partial \pi''}{\partial x} = 0, \tag{2.21}$$

$$\frac{\partial w''}{\partial t} + c_p T_{00} \left(\frac{\partial}{\partial z} + \frac{1}{2H} - \frac{1}{H_\theta}\right) \pi'' = b'', \tag{2.22}$$

$$\frac{\partial b''}{\partial t} + N^2 w'' = 0, \tag{2.23}$$

$$c_p T_{00} \frac{\partial \pi''}{\partial t} - g w'' + c_s^2 \left(\nabla \cdot \boldsymbol{v}'' + \frac{w''}{2H}\right) = 0. \tag{2.24}$$

Since all of their coefficients are constants they admit wave solutions of the form

$$\begin{pmatrix} \boldsymbol{v}'' \\ b'' \\ \pi'' \end{pmatrix} = \begin{pmatrix} \tilde{\boldsymbol{v}} \\ \tilde{b} \\ \tilde{\pi} \end{pmatrix} \exp[\mathrm{i}\,(kx + mz - \omega t)]. \tag{2.25}$$

One finds that nontrivial solutions must satisfy the dispersion relation

$$\omega^2 = \frac{c_s^2}{2}\left(k^2 + m^2 + \frac{1}{4H^2}\right) \pm \left\{\left[\frac{c_s^2}{2}\left(k^2 + m^2 + \frac{1}{4H^2}\right)\right]^2 - c_s^2 k^2 N^2\right\}^{1/2}. \tag{2.26}$$

In the limit $[(c_s^2/2)(k^2+m^2+1/4H^2)]^2 \gg c_s^2 k^2 N^2$, one recognizes the classical solutions for GWs,

$$\omega^2 = \frac{N^2 k^2}{k^2 + m^2 + \dfrac{1}{4H^2}}, \tag{2.27}$$

and sound waves,

$$\omega^2 = c_s^2 \left(k^2 + m^2 + \frac{1}{4H^2}\right). \tag{2.28}$$

Important in the present context are the polarization relations

$$\tilde{u} = \mathrm{i}\frac{k}{\omega} \frac{c_s^2 \left[m + \dfrac{\mathrm{i}}{H}\left(\dfrac{c_v}{c_p} - \dfrac{1}{2}\right)\right]}{N^2 \left(1 - \dfrac{c_s^2 k^2}{\omega^2}\right)} \tilde{b}, \tag{2.29}$$

$$\tilde{w} = \mathrm{i}\frac{\omega}{N^2}\tilde{b}, \tag{2.30}$$

$$\tilde{\pi} = \mathrm{i}\frac{c_s^2 \left[m + \dfrac{\mathrm{i}}{H}\left(\dfrac{c_v}{c_p} - \dfrac{1}{2}\right)\right]}{c_p T_{00} N^2 \left(1 - \dfrac{c_s^2 k^2}{\omega^2}\right)} \tilde{b}. \tag{2.31}$$

For typical GWs one can safely assume that

$$m \gg \frac{1}{H}, \tag{2.32}$$

$$\frac{c_s^2 k^2}{\omega^2} \gg 1, \tag{2.33}$$

yielding the approximate polarization relations

$$\tilde{u} \approx -\mathrm{i}\frac{m}{k}\frac{\omega}{N^2}\tilde{b}, \tag{2.34}$$

$$\tilde{w} \approx \mathrm{i}\frac{\omega}{N^2}\tilde{b}, \tag{2.35}$$

$$\tilde{\pi} \approx -\mathrm{i}\frac{\omega^2}{N^2}\frac{m/c_p T_{00}}{k^2}\tilde{b}. \tag{2.36}$$

Given the vertical and the horizontal scale of a wave, the dispersion relation determines the time scale. Given the scale of one of the dynamical fields, we can obtain those of the others from the polarization relations.

## 3. Scale asymptotics of the Euler equations under conditions favourable to gravity wave breaking

### 3.1. *Scale analysis*

Now consider the scales of the nonlinear dynamics of a GW propagating through an atmosphere at rest. The horizontal spatial scale, the time scale and the velocity scale are set exclusively by the wave. The vertical spatial scaling as well as that of all thermodynamic fields must also take the background atmosphere into account. Before treating the case of hydrostatic GWs with considerably longer horizontal than vertical scale, here we first focus on non-hydrostatic GWs with $m$ and $k$ being of the same order of magnitude, so that we assume

$$x = \mathscr{L}\hat{x}, \tag{3.1}$$

$$z = \mathscr{L}\hat{z}, \tag{3.2}$$

where $\mathscr{L}$ is the inverse of a typical wavenumber $K = 1/\mathscr{L}$. Likewise we introduce a typical frequency $\Omega$ and corresponding time scale $\mathscr{T} = 1/\Omega$, so that

$$t = \mathscr{T}\hat{t}. \tag{3.3}$$

We also assume that frequency scale and wavenumber scale are approximately related by the GW dispersion relation (2.27). In the non-hydrostatic limit

$$K \gg 1/2H, \tag{3.4}$$

this leads to

$$\Omega = N = \frac{g}{\sqrt{c_p T_{00}}}. \tag{3.5}$$

Note that we use the isothermal Brunt–Väisälä frequency. For the present purposes this is appropriate because we are only interested in a rough time scale estimate, and because at those heights in the atmosphere where GW breaking tends to occur the true Brunt–Väisälä frequency and its isothermal approximation are of the same order of magnitude. Referring to the conditions of the background atmosphere, we also introduce the non-dimensionalizations

$$\pi = \Pi\hat{\pi}, \tag{3.6}$$

$$\theta = \Theta\hat{\theta}, \tag{3.7}$$

where, provided $z_{00}$ is defined to be close to the breaking altitude, reasonable scales for Exner pressure and potential temperature are

$$\Pi = 1, \tag{3.8}$$

$$\Theta = T_{00.} \tag{3.9}$$

At least for the velocity scaling one must consider the dynamical wave fields. If we want to use the polarization relations (2.34)–(2.36) for fixing their scales, one of these has to be obtained independently. A critical question now is what this one scale, e.g. of the buoyancy field, can be. Since the most interesting nonlinear dynamics of GWs occurs when they are close to breaking, we focus on the specific regime when at least locally the buoyancy gradient due to the wave can neutralize that of the background atmosphere, thus enabling a static instability. It is given by

$$|\tilde{b}| = \frac{N^2}{|m|}, \tag{3.10}$$

as long as $|m| \gg 1/2H$ holds, which is guaranteed by (3.4). A reasonable buoyancy scaling for GWs can thus be expected to be

$$b' = B_w \hat{b}, \tag{3.11}$$

with

$$B_w = \frac{N^2}{K}. \tag{3.12}$$

There can be no doubt that this is a rather coarse estimate of the threshold amplitude at which an instability can set in. It is known that GWs typically become unstable at lower amplitudes (e.g. Fritts *et al.* 2006; Achatz 2007). Still, the critical wave amplitude is not less than, say, half of the value just given. Moreover, one must not forget that the estimate is from linear theory, and nonlinear dynamics changes the picture. But again, we are only interested in orders of magnitude, so the wave buoyancy scale $B_w$ suffices for our purposes. Now referring back to (2.34)–(2.35), one sees that, provided $z_{00}$ is defined to be close to the breaking altitude, velocity can be non-dimensionalized as

$$\boldsymbol{v} = \mathscr{U}\hat{\boldsymbol{v}}, \tag{3.13}$$

with velocity scale

$$\mathscr{U} = \frac{\Omega}{N^2} B_w = \frac{\Omega}{K} = \frac{\mathscr{L}}{\mathscr{T}}, \tag{3.14}$$

which turns out to also be an advective time scale. Note that the corresponding Mach number $M = \mathscr{U}/c_s$ is such that

$$\left.\begin{aligned} M^2 &= \frac{\mathscr{U}^2}{\gamma R T_{00}}, \\ &= \frac{1-\kappa}{\kappa}\epsilon^2. \end{aligned}\right\} \tag{3.15}$$

Here

$$\epsilon = \frac{\mathscr{L}}{H_\theta} = \kappa \frac{\mathscr{L}}{H} \tag{3.16}$$

is typically a small number. Again we note that here $H_\theta$ is the isothermal potential-temperature scale height, which is, however, of the same order of magnitude as the true potential-temperature scale height throughout the middle atmosphere, and also over most of the troposphere. Similarly, referring to (2.36) one sees that the wave Exner pressure can be non-dimensionalized by

$$\pi' = \Pi_w \hat{\pi}, \tag{3.17}$$

with wave Exner-pressure scale

$$\left.\begin{aligned} \Pi_w &= \frac{\Omega^2}{N^2}\frac{K/c_p T_{00}}{K^2} B_w, \\ &= \epsilon^2. \end{aligned}\right\} \tag{3.18}$$

The definition of buoyancy yields

$$\theta' = O(\Theta_w), \tag{3.19}$$

with

$$\left.\begin{aligned} \Theta_w &= \frac{T_{00}}{g} B_w, \\ &= T_{00}\epsilon. \end{aligned}\right\} \tag{3.20}$$

Note that $\Theta_w/\Theta = \epsilon$ so the potential-temperature fluctuations due to the wave are $O(\epsilon)$, while those of the Exner-pressure fluctuations are $O(\epsilon^2)$. The smallness of the Exner pressure fluctuations justifies the attempt to find sound-proof equations for GW dynamics.

We finally insert the non-dimensionalizations (3.1)–(3.3), (3.6), (3.7) and (3.13) into the Euler equations (2.1)–(2.4), and obtain

$$\epsilon^2 \frac{\mathrm{D}\hat{u}}{\mathrm{D}\hat{t}} + \hat{\theta}\frac{\partial \hat{\pi}}{\partial \hat{x}} = 0, \tag{3.21}$$

$$\epsilon^2 \frac{\mathrm{D}\hat{w}}{\mathrm{D}\hat{t}} + \hat{\theta}\frac{\partial \hat{\pi}}{\partial \hat{z}} = -\epsilon, \tag{3.22}$$

$$\frac{\mathrm{D}\hat{\theta}}{\mathrm{D}\hat{t}} = 0, \tag{3.23}$$

$$\frac{\mathrm{D}\hat{\pi}}{\mathrm{D}\hat{t}} + \frac{\kappa}{1-\kappa}\hat{\pi}\hat{\nabla}\cdot\hat{\boldsymbol{v}} = 0. \tag{3.24}$$

A major gain of the procedure just described is that we obtain one single small parameter $\epsilon$ in the non-dimensional equations. The Froude number $Fr = \mathscr{U}/\sqrt{g/K}$ satisfies

$$Fr^2 = \epsilon, \tag{3.25}$$

so that it can no longer be chosen independently from the Mach number.

### 3.2. *Scale asymptotics*

From the considerations above it follows that $\mathscr{L}$ is not the only spatial scale to be considered. Both $H$ and $H_\theta$, taken to be roughly of the same order of magnitude, also characterize relevant spatial dependence in the vertical, both of the background atmosphere and of the wave fields. The ratio between the two scales $\mathscr{L}$ and $H_\theta$, however, is $\epsilon$, so we use the ansatz

$$\begin{pmatrix} \hat{\boldsymbol{v}} \\ \hat{\theta} \\ \hat{\pi} \end{pmatrix} = \sum_{i=0}^{\infty} \epsilon^i \begin{pmatrix} \hat{\boldsymbol{v}}^{(i)} \\ \hat{\theta}^{(i)} \\ \hat{\pi}^{(i)} \end{pmatrix} (\hat{\boldsymbol{x}}, \hat{t}, \zeta), \tag{3.26}$$

where

$$\zeta = \epsilon \hat{z} \tag{3.27}$$

is a compressed vertical coordinate. From the above scale analysis, we anticipate that $\hat{\theta}^{(0)}$ and $\hat{\pi}^{(0)}$ represent the reference atmosphere, which is not supposed to have any other than large-scale dependence, so that there is no dependence on $\hat{x}$ and $\hat{z}$. In the following treatment we will see, however, that this need only be assumed for $\hat{\theta}^{(0)}$, i.e.

$$\frac{\partial \hat{\theta}^{(0)}}{\partial \hat{x}} = 0, \tag{3.28}$$

$$\frac{\partial \hat{\theta}^{(0)}}{\partial \hat{z}} = 0, \tag{3.29}$$

whereas it will be a consequence of the leading-order vertical momentum balance for $\hat{\pi}^{(0)}$. The expansion (3.26) will now be inserted into the non-dimensional equations and we will gather equal powers in $\epsilon$.

3.2.1. *Momentum equations*

Assuming non-zero $\hat{\theta}^{(0)}$, the $O(1)$ terms of the two momentum equations (3.21) and (3.22) yield

$$\frac{\partial \hat{\pi}^{(0)}}{\partial \hat{x}} = 0, \tag{3.30}$$

$$\frac{\partial \hat{\pi}^{(0)}}{\partial \hat{z}} = 0, \tag{3.31}$$

i.e. spatially the leading-order Exner pressure can only depend on the compressed vertical coordinate. As anticipated, it does not have any wave contributions. With this result, the $O(\epsilon)$ of the horizontal momentum equation (3.21) leads to

$$\frac{\partial \hat{\pi}^{(1)}}{\partial \hat{x}} = 0. \tag{3.32}$$

Thus $\pi^{(1)}$ cannot be part of the wave either. As $O(\epsilon)$ of the vertical momentum equation (3.22) one obtains, using (3.31),

$$\frac{\partial \hat{\pi}^{(1)}}{\partial \hat{z}} = -\frac{\partial \hat{\pi}^{(0)}}{\partial \zeta} - \frac{1}{\hat{\theta}^{(0)}}. \tag{3.33}$$

Using (3.29) and (3.31), this can be integrated in $\hat{z}$, yielding

$$\frac{\left[\hat{\pi}^{(1)}\right]_{\hat{z}_1}^{\hat{z}_2}}{\hat{z}_2 - \hat{z}_2} = -\frac{\partial \hat{\pi}^{(0)}}{\partial \zeta} - \frac{1}{\hat{\theta}^{(0)}}. \tag{3.34}$$

Taking the limit $|\hat{z}_2 - \hat{z}_1| \longrightarrow \infty$ and assuming sublinear growth of $\hat{\pi}^{(1)}$ in $\hat{z}$, one obtains

$$\frac{\partial \hat{\pi}^{(0)}}{\partial \zeta} = -\frac{1}{\hat{\theta}^{(0)}}. \tag{3.35}$$

This is simply hydrostatic equilibrium of the reference atmosphere. This, inserted into (3.33), also yields

$$\frac{\partial \hat{\pi}^{(1)}}{\partial \hat{z}} = 0, \tag{3.36}$$

which supplements (3.32). Finally, to $O(\epsilon^2)$ one obtains from the horizontal momentum equation, using (3.30) and (3.32),

$$\frac{D_0 \hat{u}^{(0)}}{D\hat{t}} + \hat{\theta}^{(0)} \frac{\partial \hat{\pi}^{(2)}}{\partial \hat{x}} = 0, \tag{3.37}$$

with the definition

$$\frac{D_0}{D\hat{t}} = \frac{\partial}{\partial \hat{t}} + \hat{u}^{(0)} \frac{\partial}{\partial \hat{x}} + \hat{w}^{(0)} \frac{\partial}{\partial \hat{z}}. \tag{3.38}$$

Similarly, the vertical momentum equation yields, with the help of (3.35),

$$\frac{D_0 \hat{w}^{(0)}}{D\hat{t}} + \hat{\theta}^{(0)} \left( \frac{\partial \hat{\pi}^{(2)}}{\partial \hat{z}} + \frac{\partial \hat{\pi}^{(1)}}{\partial \zeta} \right) = \frac{\hat{\theta}^{(1)}}{\hat{\theta}^{(0)}}. \tag{3.39}$$

3.2.2. *Entropy equation*

Turning now to the entropy equation (3.23), we obtain to $O(1)$

$$\frac{D_0 \hat{\theta}^{(0)}}{D\hat{t}} = 0. \tag{3.40}$$

Together with the basic assumptions (3.28) and (3.29), this yields

$$\frac{\partial \hat{\theta}^{(0)}}{\partial \hat{t}} = 0, \tag{3.41}$$

i.e. the reference potential temperature is independent of time. Finally, the $O(\epsilon)$ of the entropy equation gives, again using (3.28) and (3.29),

$$\frac{D_0 \hat{\theta}^{(1)}}{D\hat{t}} + \hat{w}^{(0)} \frac{\partial \hat{\theta}^{(0)}}{\partial \zeta} = 0. \tag{3.42}$$

3.2.3. *Exner-pressure equation*

To $O(1)$, one obtains from the Exner-pressure equation (3.24), using (3.30) and (3.31),

$$\frac{\partial \hat{\pi}^{(0)}}{\partial \hat{t}} + \frac{\kappa}{1-\kappa} \hat{\pi}^{(0)} \left( \frac{\partial \hat{u}^{(0)}}{\partial \hat{x}} + \frac{\partial \hat{w}^{(0)}}{\partial \hat{z}} \right) = 0. \tag{3.43}$$

Integrating over an arbitrary volume $\hat{V}$ in $\hat{x}$ and $\hat{z}$ yields

$$\frac{1}{\hat{\pi}^{(0)}} \frac{\partial \hat{\pi}^{(0)}}{\partial \hat{t}} + \frac{\kappa}{1-\kappa} \frac{1}{\hat{V}} \oint_{\hat{V}} \hat{\boldsymbol{v}}^{(0)} \cdot \mathrm{d}\hat{\boldsymbol{S}}, \tag{3.44}$$

where we have used the Gauss integration theorem, with self-evident notation. By taking the limit $\hat{V} \longrightarrow \infty$, and again applying the sublinear growth condition, but now for $\hat{\boldsymbol{v}}^{(0)}$, we obtain

$$\frac{1}{\hat{\pi}^{(0)}} \frac{\partial \hat{\pi}^{(0)}}{\partial \hat{t}} = 0. \tag{3.45}$$

This supplements (3.40), i.e. the background atmosphere does not depend on time. Substituting this back into (3.43) one also obtains

$$\frac{\partial \hat{u}^{(0)}}{\partial \hat{x}} + \frac{\partial \hat{w}^{(0)}}{\partial \hat{z}} = 0. \tag{3.46}$$

To leading order, the velocity field is non-divergent. One might be tempted to stop here, and conclude that GWs near their breaking altitude are to be described by the sound-proof Boussinesq equations. Indeed this is a fruitful approach. If, however, one is also interested in incorporating wave growth due to ambient density gradients, one inevitably must go to the next order $O(\epsilon)$ of the Exner-pressure equation. Using (3.30)–(3.32), (3.36) and (3.46), this is

$$\frac{\partial \hat{\pi}^{(1)}}{\partial \hat{t}} + \hat{w}^{(0)} \frac{\partial \hat{\pi}^{(0)}}{\partial \zeta} + \frac{\kappa}{1-\kappa} \hat{\pi}^{(0)} \left( \frac{\partial \hat{u}^{(1)}}{\partial \hat{x}} + \frac{\partial \hat{w}^{(1)}}{\partial \hat{z}} + \frac{\partial \hat{w}^{(0)}}{\partial \zeta} \right) = 0. \tag{3.47}$$

At first impression this is a predictive equation for $\hat{\pi}^{(1)}$. We offer, however, two arguments why one can safely assume

$$\frac{\partial \hat{\pi}^{(1)}}{\partial \hat{t}} = 0. \tag{3.48}$$

(*a*) Up to the first term in (3.47), all others are linear in the wave velocity field. If one assumes that all wave velocity fields are fluctuating so that their volume integral does not diverge, i.e.

$$\lim_{\hat{V} \to \infty} \frac{1}{\hat{V}} \int_{\hat{V}} \mathrm{d}\hat{V} \, \hat{\boldsymbol{v}}^{(0)} = 0, \tag{3.49}$$

$$\lim_{\hat{V} \to \infty} \frac{1}{\hat{V}} \int_{\hat{V}} \mathrm{d}\hat{V} \, \hat{\boldsymbol{v}}^{(1)} = 0, \tag{3.50}$$

then, integrating (3.47) accordingly, and using (3.32) and (3.36), one obtains (3.48).

(*b*) The linear theory for GWs at saturation amplitude discussed above yields Exner-pressure fluctuations which are at lowest order $O(\epsilon^2)$. Thus one can assume

$$\hat{\pi}^{(1)} = 0, \tag{3.51}$$

which also clearly leads to (3.48).

The first-order contribution to the Exner pressure, if it exists at all, is thus time-independent. Since it also does not depend on $\hat{x}$ and $\hat{z}$ one might interpret it as part of the Exner pressure of the reference atmosphere. In principle, one could try and absorb it into $\hat{\pi}^{(0)}$, but then the hydrostatic equilibrium (3.35) would not hold any more. Finally, substituting (3.48) into (3.47) yields

$$\hat{w}^{(0)} \frac{\partial \hat{\pi}^{(0)}}{\partial \zeta} + \frac{\kappa}{1-\kappa} \hat{\pi}^{(0)} \left( \frac{\partial \hat{u}^{(1)}}{\partial \hat{x}} + \frac{\partial \hat{w}^{(1)}}{\partial \hat{z}} + \frac{\partial \hat{w}^{(0)}}{\partial \zeta} \right) = 0. \tag{3.52}$$

With (3.37), (3.39), (3.42), (3.46) and (3.51) we have the leading-order closed predictive system, namely the classical Boussinesq approximation for small-scale flows in the vicinity of some given reference height $z_{00}$. This system, however, does not describe the amplification of internal waves as they move vertically over distances comparable to $H$ in the atmosphere. The first-order correction to the leading-order divergence constraint as given in (3.52) shows that the effect responsible for this wave amplification appears only at the next order in the asymptotic expansion. Thus, we will employ methods of multiple-scale analysis to systematically describe this process in §§ 6 and 7 below.

## 4. Scale asymptotics of the sound-proof equation systems

Before we do so, we will demonstrate in § 4 that the pseudo-incompressible equations are consistent with the scale asymptotics for the full compressible system including the first-order divergence constraint, whereas the anelastic equations are not, unless the background stratification is small so that $H_\theta \gg H$.

### 4.1. *The pseudo-incompressible equations*

The pseudo-incompressible equations (Durran 1989) can be written as

$$\frac{\mathrm{D}u}{\mathrm{D}t} + c_p\theta\frac{\partial \pi}{\partial x} = 0, \tag{4.1}$$

$$\frac{\mathrm{D}w}{\mathrm{D}t} + c_p\theta\frac{\partial \pi}{\partial z} = -g, \tag{4.2}$$

$$\frac{\mathrm{D}\theta}{\mathrm{D}t} = 0, \tag{4.3}$$

$$\nabla\cdot(\overline{\rho}\overline{\theta}\boldsymbol{v}) = 0, \tag{4.4}$$

where the prescribed reference state is assumed to be hydrostatic, i.e. it satisfies (2.7). Moreover, it satisfies (2.6), or rather

$$\overline{\rho}\overline{\theta} = \frac{p_{00}}{R}\overline{\pi}^{(1-\kappa)/\kappa}, \tag{4.5}$$

so that (4.4) also takes the form

$$\nabla\cdot\left(\overline{\pi}^{(1-\kappa)/\kappa}\boldsymbol{v}\right) = 0, \tag{4.6}$$

or rather

$$w\frac{\partial\overline{\pi}}{\partial z} + \frac{\kappa}{1-\kappa}\overline{\pi}\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) = 0, \tag{4.7}$$

which will be used here. The total thermodynamic fields include the reference state, so one can write

$$\theta = \overline{\theta} + \theta', \tag{4.8}$$

$$\pi = \overline{\pi} + \pi'. \tag{4.9}$$

Consistent with the above we now assume the same scaling, i.e. we non-dimensionalize the pseudo-incompressible equations using (3.1)–(3.3), (3.6), (3.7) and (3.13), and expand the non-dimensional fields as in (3.26). The reference atmosphere is taken to be represented by the zero-order expansion. In other words, we assume

$$\begin{pmatrix}\overline{\theta}\\ \overline{\pi}\end{pmatrix} = \begin{pmatrix}\Theta\hat{\theta}^{(0)}\\ \Pi\hat{\pi}^{(0)}\end{pmatrix}(\zeta), \tag{4.10}$$

$$\begin{pmatrix}\theta'\\ \pi'\end{pmatrix} = \begin{pmatrix}\Theta\hat{\theta}'\\ \Pi\hat{\pi}'\end{pmatrix}(\hat{\boldsymbol{x}},\hat{t},\zeta) = \sum_{i=1}^{\infty}\epsilon^i\begin{pmatrix}\Theta\hat{\theta}^{(i)}\\ \Pi\hat{\pi}^{(i)}\end{pmatrix}(\hat{\boldsymbol{x}},\hat{t},\zeta). \tag{4.11}$$

We assume *a priori* that the zero-order thermodynamic fields only depend on $\zeta$, as is consistent with the findings above. The velocity field is expanded as in (3.26):

$$\boldsymbol{v} = \mathscr{U}\sum_{i=0}^{\infty}\epsilon^i\hat{\boldsymbol{v}}^{(i)}(\hat{\boldsymbol{x}},\hat{t},\zeta). \tag{4.12}$$

The momentum equations in the compressible Euler system and the pseudo-incompressible system agree completely. Therefore, it need not be shown that a scale-asymptotic analysis of these yields the same results. More specifically, while (3.30) and (3.31) are satisfied by assumption, (3.32), (3.35)–(3.37) and (3.39) follow from the analysis. The same holds for the analysis of the entropy equation. Both (3.41) and (3.42) are obtained. Only the comparison between the asymptotics of the Exner-pressure equation (2.4) and the divergence condition (4.7) requires some consideration. Inserting (4.10) and (4.12) into (4.7) yields to $O(1)$ the non-divergence condition (3.46), while (3.45) is satisfied by basic assumption. Finally, the $O(\epsilon)$-terms of (4.7) yield (3.52).

The only result one cannot obtain is (3.48), i.e. the time-independence of $\hat{\pi}^{(1)}$. This, however, is no real problem since we are interested in GW scaling, and can thus assume (3.51). Thus motivated we will use in the remainder of this study that, consistent with GW scaling, there is no $O(\epsilon)$ contribution to the Exner pressure.

### 4.2. *The anelastic equations*

The anelastic equations (Lipps & Hemler 1982) can be written

$$\frac{\mathrm{D}u}{\mathrm{D}t} + \frac{\partial}{\partial x}\left(\frac{p'}{\overline{\rho}}\right) = 0, \tag{4.13}$$

$$\frac{\mathrm{D}w}{\mathrm{D}t} + \frac{\partial}{\partial z}\left(\frac{p'}{\overline{\rho}}\right) = g\frac{\theta'}{\overline{\theta}}, \tag{4.14}$$

$$\frac{\mathrm{D}\theta}{\mathrm{D}t} = 0, \tag{4.15}$$

$$\nabla\cdot(\overline{\rho}\boldsymbol{v}) = 0, \tag{4.16}$$

where the prescribed reference state is the same as above. The deviatoric pressure and potential temperature are $p' = p - \overline{p}$ and $\theta' = \theta - \overline{\theta}$. From the definitions one finds

$$\frac{p'}{\overline{\rho}} = R\frac{\overline{\theta}}{\overline{\pi}^{(1-\kappa)/\kappa}}(\pi^{1/\kappa} - \overline{\pi}^{1/\kappa}). \tag{4.17}$$

The thermodynamic fields are split and scaled as in (3.51) and (4.8)–(4.11). Non-dimensionalizing then yields

$$\epsilon^2\frac{\mathrm{D}\hat{u}}{\mathrm{D}\hat{t}} + \kappa\frac{\partial}{\partial\hat{x}}\left\{\frac{\hat{\theta}^{(0)}}{\hat{\pi}^{(0)\,(1-\kappa)/\kappa}}\left[\left(\hat{\pi}^{(0)} + \hat{\pi}'\right)^{1/\kappa} - \hat{\pi}^{(0)\,1/\kappa}\right]\right\} = 0, \tag{4.18}$$

$$\epsilon^2\frac{\mathrm{D}\hat{w}}{\mathrm{D}\hat{t}} + \kappa\frac{\partial}{\partial\hat{z}}\left\{\frac{\hat{\theta}^{(0)}}{\hat{\pi}^{(0)\,(1-\kappa)/\kappa}}\left[\left(\hat{\pi}^{(0)} + \hat{\pi}'\right)^{1/\kappa} - \hat{\pi}^{(0)\,1/\kappa}\right]\right\} = \epsilon\frac{\hat{\theta}'}{\hat{\theta}^{(0)}}, \tag{4.19}$$

$$\frac{\mathrm{D}\hat{\theta}'}{\mathrm{D}\hat{t}} + \epsilon\hat{w}\frac{\mathrm{d}\hat{\theta}^{(0)}}{\mathrm{d}\zeta} = 0, \tag{4.20}$$

$$\frac{\kappa}{1-\kappa}\hat{\pi}^{(0)}\hat{\nabla}\cdot\hat{\boldsymbol{v}} + \epsilon\hat{w}\left(\frac{\partial\hat{\pi}^{(0)}}{\partial\zeta} - \frac{\kappa}{1-\kappa}\frac{\hat{\pi}^{(0)}}{\hat{\theta}^{(0)}}\frac{\partial\hat{\theta}^{(0)}}{\partial\zeta}\right) = 0. \tag{4.21}$$

The leading order $O(\epsilon^2)$ of the horizontal momentum equation gives exactly the result (3.37) from the Euler equations. From the vertical momentum equation one obtains to the same order

$$\frac{\mathrm{D}_0\hat{w}^{(0)}}{\mathrm{D}\hat{t}} + \hat{\theta}^{(0)}\frac{\partial\hat{\pi}^{(2)}}{\partial\hat{z}} = \frac{\hat{\theta}^{(1)}}{\hat{\theta}^{(0)}}, \tag{4.22}$$

which agrees with the corresponding result (3.39) from the Euler equations in the assumed case $\hat{\pi}^{(1)} = 0$, consistent with GW scaling. Also, to leading order $O(\epsilon)$ the anelastic entropy equation is consistent with the Euler-equation result (3.42). What remains is the anelastic continuity equation (4.21). To leading order $O(1)$, it yields the same flow non-divergence (3.46) as obtained from the Euler equations. A difference arises, however, in the next order $O(\epsilon)$, from which we get

$$\hat{w}^{(0)} \frac{\partial \hat{\pi}^{(0)}}{\partial \zeta} + \frac{\kappa}{1-\kappa} \hat{\pi}^{(0)} \left( \frac{\partial \hat{u}^{(1)}}{\partial \hat{x}} + \frac{\partial \hat{w}^{(1)}}{\partial \hat{z}} + \frac{\partial \hat{w}^{(0)}}{\partial \zeta} \right) - \frac{\kappa}{1-\kappa} \frac{\hat{\pi}^{(0)}}{\hat{\theta}^{(0)}} \hat{w}^{(0)} \frac{\partial \hat{\theta}^{(0)}}{\partial \zeta} = 0. \tag{4.23}$$

In comparison to the Euler-equation result (3.52), we are left with an additional term of relative magnitude $\kappa/(1-\kappa)\,|\mathrm{d}\hat{\theta}^{(0)}/\mathrm{d}\zeta| = R/c_v\,|\mathrm{d}\hat{\theta}^{(0)}/\mathrm{d}\zeta|$. We conclude that the pseudo-incompressible equations are consistent with the Euler equations in the description of non-hydrostatic GWs, while the anelastic equations are consistent only if $R/c_v\, 1/\hat{\theta}^{(0)}\,|\mathrm{d}\hat{\theta}^{(0)}/\mathrm{d}\zeta| \ll 1/\hat{\pi}^{(0)}|\mathrm{d}\hat{\pi}^{(0)}/\mathrm{d}\zeta|$, i.e. if the potential-temperature scale height is much larger than that of the Exner pressure multiplied by $R/c_v$. Especially in the stratosphere and the mesosphere, however, this is typically not the case.

## 5. Hydrostatic gravity waves

So far, we have assumed non-hydrostatic GWs with equal spatial scales in the horizontal and the vertical directions, respectively. Here we show that the consistency between the scale asymptotics of the Euler equations and the pseudo-incompressible equations also holds for hydrostatic GWs, in fact also potentially influenced by rotation, here deliberately neglected for simplicity, with a longer horizontal than vertical scale. For this purpose, we stick with the spatial non-dimensionalizations (3.1) and (3.2), but introduce a compressed horizontal coordinate

$$\chi = \epsilon \hat{x}, \tag{5.1}$$

and assume that all fields depend on $\chi$ instead of $\hat{x}$. In other words, we assume the horizontal and short-vertical spatial scale to have a ratio $\epsilon$. In terms of wavenumbers this means

$$k = O(\epsilon K), \tag{5.2}$$

$$m = O(K). \tag{5.3}$$

As long as $K \gg 1/2H$, the dispersion relation (2.27) then yields

$$\omega = O(\epsilon N). \tag{5.4}$$

Thus, the relevant time scale is by a factor $\epsilon$ longer than assumed in the non-dimensionalization (3.3), so we also introduce a compressed time coordinate

$$\tau = \epsilon \hat{t}, \tag{5.5}$$

and assume that all fields depend on $\tau$ instead of $\hat{t}$.

With regard to the wave fields, first note that the instability criterion (3.10) is the same as for non-hydrostatic GWs, so $B_w$ is still the correct scaling for the wave buoyancy. In the absence of rotation even hydrostatic waves cannot be destabilized by their intrinsic shear, while this would be the case otherwise. Then the stability criterion would have to be modified by considering the Richardson number (Dunkerton 1997). Here, however, the polarization relations (2.34) and (2.35), using (5.2)–(5.4), then yield

$$u = \mathscr{U} O(1), \tag{5.6}$$

$$w = \mathscr{U} O(\epsilon). \tag{5.7}$$

Similarly, one obtains from (2.36) for the wave part of the Exner pressure

$$\pi' = \Pi O(\epsilon^2). \tag{5.8}$$

### 5.1. *Scale asymptotics of the Euler equations*

Therefore, we assume the following expansions, non-dimensionalizing wind, potential temperature and Exner pressure as before by $U$, $T_{00}$ and $\Pi$, respectively:

$$\hat{u} = \sum_{i=0}^{\infty} \epsilon^i \hat{u}^{(i)}(\chi, \hat{z}, \zeta, \tau), \tag{5.9}$$

$$\hat{w} = \sum_{i=1}^{\infty} \epsilon^i \hat{w}^{(i)}(\chi, \hat{z}, \zeta, \tau), \tag{5.10}$$

$$\hat{\theta} = \theta^{(0)}(\zeta, \tau) + \sum_{i=1}^{\infty} \epsilon^i \hat{\theta}^{(i)}(\chi, \hat{z}, \zeta, \tau), \tag{5.11}$$

$$\hat{\pi} = \pi^{(0)}(\chi, \hat{z}, \zeta, \tau) + \sum_{i=2}^{\infty} \epsilon^i \hat{\pi}^{(i)}(\chi, \hat{z}, \zeta, \tau). \tag{5.12}$$

Note that we assume *a priori* that $\hat{\pi}^{(1)} = 0$. Substituting this into the Euler equations we again order by equal powers in $\epsilon$. Most of the treatment is very much analogous to that of the non-hydrostatic situation so only the main steps will be given here.

#### 5.1.1. *Momentum equations*

The $O(1)$ terms in the vertical momentum equation yield

$$\frac{\partial \hat{\pi}^{(0)}}{\partial \hat{z}} = 0. \tag{5.13}$$

The lowest non-trivial order in the horizontal momentum equation is $O(\epsilon)$. It leads to

$$\frac{\partial \hat{\pi}^{(0)}}{\partial \chi} = 0. \tag{5.14}$$

Similarly the same order in the vertical momentum equation gives, using (5.13), the hydrostatic equilibrium

$$\frac{\partial \hat{\pi}^{(0)}}{\partial \zeta} = -\frac{1}{\hat{\theta}^{(0)}}. \tag{5.15}$$

To next order, using (5.14), the horizontal momentum equation turns out to be trivial, while, using (5.13) and (5.15), the vertical momentum equation yields

$$\hat{\theta}^{(0)} \frac{\partial \hat{\pi}^{(2)}}{\partial \hat{z}} = \frac{\hat{\theta}^{(1)}}{\hat{\theta}^{(0)}}, \tag{5.16}$$

so that the wave pressure is to leading order indeed in hydrostatic equilibrium. Time derivatives appear no sooner than to order $O(\epsilon^3)$. The horizontal momentum equation yields

$$\frac{\partial \hat{u}^{(0)}}{\partial \tau} + \hat{u}^{(0)} \frac{\partial \hat{u}^{(0)}}{\partial \chi} + \hat{w}^{(1)} \frac{\partial \hat{u}^{(0)}}{\partial \hat{z}} + \hat{\theta}^{(0)} \frac{\partial \hat{\pi}^{(2)}}{\partial \chi} = 0, \tag{5.17}$$

while one obtains from the vertical momentum equation, using (5.13), (5.15) and (5.16),

$$\hat{\theta}^{(0)} \left( \frac{\partial \hat{\pi}^{(3)}}{\partial \hat{z}} + \frac{\partial \hat{\pi}^{(2)}}{\partial \zeta} \right) = - \left[ \frac{\hat{\theta}^{(1)}}{\hat{\theta}^{(0)}} \right]^2 + \frac{\hat{\theta}^{(2)}}{\hat{\theta}^{(0)}}. \tag{5.18}$$

This is an extension of the hydrostatic equilibrium of the wave fields to next order over (5.16).

5.1.2. *Entropy equation*

The lowest non-trivial order of the entropy equation is $O(\epsilon)$. It yields time-independence of the potential temperature of the reference atmosphere:

$$\frac{\partial \hat{\theta}^{(0)}}{\partial \tau} = 0. \tag{5.19}$$

To next order we obtain

$$\frac{\partial \hat{\theta}^{(1)}}{\partial \tau} + \hat{u}^{(0)} \frac{\partial \hat{\theta}^{(1)}}{\partial \chi} + \hat{w}^{(1)} \frac{\partial \hat{\theta}^{(1)}}{\partial \hat{z}} + \hat{w}^{(1)} \frac{\partial \hat{\theta}^{(0)}}{\partial \zeta} = 0. \tag{5.20}$$

5.1.3. *Exner-pressure equation*

The lowest non-trivial order of the Exner-pressure equation is $O(\epsilon)$. Together with (5.13) and (5.14) this yields

$$\frac{\partial \hat{\pi}^{(0)}}{\partial \tau} + \frac{\kappa}{1-\kappa} \hat{\pi}^{(0)} \left( \frac{\partial \hat{u}^{(0)}}{\partial \chi} + \frac{\partial \hat{w}^{(1)}}{\partial \hat{z}} \right) = 0. \tag{5.21}$$

This is the same type of equation as (3.43). In the same manner as there we assume that spatial dependence of the lowest order velocity field in $\chi$ and $\hat{z}$ does not diverge, which leads via Gauss integration to

$$\frac{\partial \hat{\pi}^{(0)}}{\partial \tau} = 0, \tag{5.22}$$

$$\frac{\partial \hat{u}^{(0)}}{\partial \chi} + \frac{\partial \hat{w}^{(1)}}{\partial \hat{z}} = 0. \tag{5.23}$$

To next order $O(\epsilon^2)$, one directly obtains

$$\hat{w}^{(1)} \frac{\partial \hat{\pi}^{(0)}}{\partial \zeta} + \frac{\kappa}{1-\kappa} \hat{\pi}^{(0)} \left( \frac{\partial \hat{u}^{(1)}}{\partial \chi} + \frac{\partial \hat{w}^{(1)}}{\partial \zeta} + \frac{\partial \hat{w}^{(2)}}{\partial \hat{z}} \right) = 0. \tag{5.24}$$

## 5.2. *Scale asymptotics of the pseudo-incompressible equations*

Consistency between the Euler equations and the pseudo-incompressible equations demands that the scale asymptotics of the latter is the same, up to all orders discussed above, for the Euler equations. Since momentum equations and entropy equation agree between the two systems, one need only to show that the scale asymptotics of the pseudo-incompressible non-divergence condition (4.7) yields, up to $O(\epsilon^2)$, (5.22)–(5.24). We assume the expansions (5.9)–(5.12), and also in addition that $\hat{\theta}^{(0)}$ and $\hat{\pi}^{(0)}$, which represent the reference atmosphere, depend exclusively on $\zeta$, just as also derived from the scale asymptotics of the Euler equations. Then (5.22) is satisfied by assumption. The lowest non-trivial order of (4.7) is $O(\epsilon)$, directly yielding (5.23). To next order we directly obtain (5.24).

## 6. Wentzel–Kramers–Brillouin theory

In § 3, we have studied the asymptotics of internal waves in a stratified compressible atmosphere under conditions favourable to wave breaking. The leading-order equations for length scales of order $O(\epsilon H_\theta)$, i.e. length scales small compared to the potential-temperature scale height, turned out to be the incompressible Boussinesq equations, while at first order we found divergence corrections that correspond to Durran's pseudo-incompressible model. In § 6, we demonstrate that these divergence corrections affect the amplification of initially weak internal waves to leading order as wavepackets travel upwards in the atmosphere to vertical levels comparable to $H_\theta$.

### 6.1. *WKB expansions*

To this end we examine the accumulation of first-order effects and how they can affect the leading-order solution over long times which we resolve by the time variable $\tau = \epsilon \hat{t}$. On this time scale, wavepackets can travel over distances of order $O(H_\theta)$, while undergoing non-trivial deformation and amplification or damping. Additionally, two different horizontal scales $x$ and $\chi$ are used in the analysis, where $x$ represents the scales of horizontal variation in the phase of the wavepacket and $\chi$ its large-scale horizontal envelope. Therefore, we construct WKB-type asymptotic multiscale solutions to the compressible flow equations from (3.21) to (3.24). Note that perturbation-energy conservation (2.15) suggests that

$$\hat{\boldsymbol{v}} \propto \overline{\rho}^{-1/2}, \tag{6.1}$$

$$\hat{\theta}' \propto \overline{\theta}\,\overline{\rho}^{-1/2}, \tag{6.2}$$

$$\hat{\pi}' \propto \overline{\theta}^{-1}\overline{\rho}^{-1/2}. \tag{6.3}$$

The amplitude of a vertically propagating wave is thus affected by both the ambient density and the ambient potential temperature. For a low-amplitude wavepacket below the breaking altitude, we thus assume

$$\hat{u} = \epsilon \tilde{u}^{(0)}, \tag{6.4}$$

$$\hat{w} = \epsilon \tilde{w}^{(0)}, \tag{6.5}$$

$$\hat{\theta} = \epsilon^{\nu}\left(\theta^{(0)} + \epsilon^2 \tilde{\theta}^{(1)}\right), \tag{6.6}$$

$$\hat{\pi} = \epsilon^{-\nu}\left(\pi^{(0)} + \epsilon^3 \tilde{\pi}^{(2)}\right), \tag{6.7}$$

so that factors $\epsilon$ and $\epsilon^{\nu}$ describe the density and potential-temperature effect, respectively. The parameter $\nu$ may be chosen accordingly, but it does not appear in the further analysis. We make the usual WKB assumption that each wave field has local amplitude, wavenumber, and frequency with only slow dependence on space and time (Bretherton 1966), i.e.

$$\begin{pmatrix} \tilde{u}^{(0)} \\ \tilde{w}^{(0)} \\ \tilde{\theta}^{(1)} \\ \tilde{\pi}^{(2)} \end{pmatrix} = \mathrm{Re}\left\{ \left[ \begin{pmatrix} \hat{U}^{(0)} \\ \hat{W}^{(0)} \\ \hat{\Theta}^{(1)} \\ \hat{\Pi}^{(2)} \end{pmatrix} (\tau, \chi, \zeta) + \epsilon \begin{pmatrix} \hat{U}^{(1)} \\ \hat{W}^{(1)} \\ \hat{\Theta}^{(2)} \\ \hat{\Pi}^{(3)} \end{pmatrix} (\tau, \chi, \zeta) \right] \exp\left( \mathrm{i}\, \frac{\varphi(\tau, \chi, \zeta)}{\epsilon} \right) \right\} + o(\epsilon). \tag{6.8}$$

Local frequency and horizontal and vertical wavenumber are defined as time derivative and gradient components of the phase, i.e.

$$\omega = -\frac{\partial}{\partial t}\left(\frac{\varphi}{\epsilon}\right) = -\frac{\partial \varphi}{\partial \tau}, \tag{6.9}$$

$$k = \frac{\partial}{\partial x}\left(\frac{\varphi}{\epsilon}\right) = \frac{\partial \varphi}{\partial \chi}, \tag{6.10}$$

$$m = \frac{\partial}{\partial z}\left(\frac{\varphi}{\epsilon}\right) = \frac{\partial \varphi}{\partial \zeta}. \tag{6.11}$$

Notice that we have not included a quadratic term involving $\exp(\mathrm{i}\,2\varphi/\epsilon)$ in the first-order terms. This anticipates the leading-order result given below, which shows the leading-order velocities to be solenoidal. Solenoidal fields produce advective tendencies no earlier than at second order in the expansions, which we will not consider here. For clarity of presentation we are also not including a mean flow. We have verified, however, that the more complete case can be dealt with along the lines outlined below for the large-amplitude case. The perturbation-energy conservation derived here is then replaced by wave-action conservation (Bretherton 1966). In the small-amplitude case, the mean flow, however, is not affected by the waves.

Noting that for every of the four fields, e.g. the horizontal wind,

$$\frac{\partial \tilde{u}^{(0)}}{\partial \hat{t}} = \mathrm{Re}\left[\left(-\mathrm{i}\omega\hat{\mathrm{U}} + \epsilon\,\frac{\partial \hat{\mathrm{U}}}{\partial \tau}\right)\exp\left(\mathrm{i}\,\frac{\varphi}{\epsilon}\right)\right] + \text{h.o.t.}, \tag{6.12}$$

$$\frac{\partial \tilde{u}^{(0)}}{\partial \hat{x}} = \mathrm{Re}\left[\left(\mathrm{i}k\hat{\mathrm{U}} + \epsilon\,\frac{\partial \hat{\mathrm{U}}}{\partial \chi}\right)\exp\left(\mathrm{i}\,\frac{\varphi}{\epsilon}\right)\right] + \text{h.o.t.}, \tag{6.13}$$

$$\frac{\partial \tilde{u}^{(0)}}{\partial \hat{z}} = \mathrm{Re}\left[\left(\mathrm{i}m\hat{\mathrm{U}} + \epsilon\,\frac{\partial \hat{\mathrm{U}}}{\partial \zeta}\right)\exp\left(\mathrm{i}\,\frac{\varphi}{\epsilon}\right)\right] + \text{h.o.t.}, \tag{6.14}$$

with $\hat{U} \equiv \widehat{U}^{(0)} + \epsilon\,\widehat{U}^{(1)}$ and h.o.t. abbreviating 'higher-order terms', we obtain to leading order from the vertical momentum equation (3.22), the hydrostatic balance (3.35), and then from (3.21)–(3.24)

$$\underbrace{\begin{pmatrix} -\mathrm{i}\omega & 0 & 0 & \mathrm{i}k \\ 0 & -\mathrm{i}\omega & -N & \mathrm{i}m \\ 0 & N & -\mathrm{i}\omega & 0 \\ \mathrm{i}k & \mathrm{i}m & 0 & 0 \end{pmatrix}}_{\boldsymbol{M}(\omega,k,m)} \begin{pmatrix} \hat{U}^{(0)} \\ \hat{W}^{(0)} \\ \dfrac{1}{N}\dfrac{\hat{\Theta}^{(1)}}{\hat{\theta}^{(0)}} \\ \hat{\theta}^{(0)}\hat{\Pi}^{(2)} \end{pmatrix} = 0. \tag{6.15}$$

Here $N^2 = 1/\hat{\theta}^{(0)}\,\mathrm{d}\hat{\theta}^{(0)}/\mathrm{d}\zeta$ is the non-dimensional squared Brunt–Väisälä frequency. The formulation in (6.15) has been chosen to explicitly reveal that the relevant matrix $\boldsymbol{M}(\omega, k, m)$ is anti-Hermitian. Therefore, only imaginary eigenvalues of $\boldsymbol{M}$ exist, which correspond to travelling wave solutions. To next order we obtain

$$\boldsymbol{M}\begin{pmatrix} \hat{U}^{(1)} \\ \hat{W}^{(1)} \\ \dfrac{1}{N}\dfrac{\hat{\Theta}^{(2)}}{\hat{\theta}^{(0)}} \\ \hat{\theta}^{(0)}\hat{\Pi}^{(3)} \end{pmatrix} = \begin{pmatrix} -\dfrac{\partial \hat{U}^{(0)}}{\partial \tau} - \hat{\theta}^{(0)}\dfrac{\partial \hat{\Pi}^{(2)}}{\partial \chi} \\ -\dfrac{\partial \hat{W}^{(0)}}{\partial \tau} - \hat{\theta}^{(0)}\dfrac{\partial \hat{\Pi}^{(2)}}{\partial \zeta} \\ -\dfrac{\partial}{\partial \tau}\left(\dfrac{\hat{\Theta}^{(1)}}{\hat{\theta}^{(0)}}\right) \\ -\dfrac{\partial \hat{U}^{(0)}}{\partial \chi} - \dfrac{\partial \hat{W}^{(0)}}{\partial \zeta} - \dfrac{1-\kappa}{\kappa}\dfrac{\hat{W}^{(0)}}{\hat{\pi}^{(0)}}\dfrac{\partial \hat{\pi}^{(0)}}{\partial \chi} \end{pmatrix}. \tag{6.16}$$

Note that we have used the solenoidality (incompressibility) condition from (6.15), i.e. $k\hat{U}^{(0)} + m\hat{W}^{(0)} = 0$, to eliminate the advection terms on the right-hand side of (6.16).

### 6.2. *Leading-order analysis*

At every $(\tau, \chi, \zeta)$, (6.15) represents a linear system of equations for the amplitudes which has a non-trivial solution only if $\det(\boldsymbol{M}) = 0$. A straightforward calculation shows that this is equivalent to

$$\omega^2(k, m) = N^2 \frac{k^2}{k^2 + m^2}. \tag{6.17}$$

Of course, (6.17) is the internal-wave dispersion relation for an incompressible Boussinesq fluid, and this is what was to be expected here, as the flow Mach number considered is $O(\epsilon)$, implying sound-proof motions, and the length scale of the internal waves, given by the inverse of the wavenumber, is $\epsilon H_\theta \ll H$, so that vertical density variations do not play a role at leading order.

Given the dispersion relation (6.17) relating $\omega$ to $k$ and $m$, thereby prescribing the phase speed of the short-internal GWs through the Hamilton–Jacobi equation

$$\frac{\partial \varphi}{\partial \tau} + \omega\left(\frac{\partial \varphi}{\partial \chi}, \frac{\partial \varphi}{\partial \zeta}\right) = 0, \tag{6.18}$$

the system matrix $\boldsymbol{M}$ of the leading-order equations becomes singular, and the vector of amplitudes has to lie in the null space of the matrix. Again we leave out the tedious, but straightforward, details in concluding that one may write the amplitude vector as

$$\begin{pmatrix} \hat{U}^{(0)} \\ \hat{W}^{(0)} \\ \dfrac{1}{N}\dfrac{\hat{\Theta}^{(1)}}{\hat{\theta}^{(0)}} \\ \hat{\theta}^{(0)}\hat{\Pi}^{(2)} \end{pmatrix} = \begin{pmatrix} -\mathrm{i}\dfrac{m}{k}\dfrac{\omega}{N} \\ \mathrm{i}\dfrac{\omega}{N} \\ 1 \\ -\mathrm{i}\dfrac{m}{k^2}\dfrac{\omega^2}{N} \end{pmatrix} \frac{1}{N}\frac{\hat{\Theta}^{(1)}}{\hat{\theta}^{(0)}}. \tag{6.19}$$

### 6.3. *First-order analysis and the evolution of the wave amplitudes*

Equations (6.19) settle the solution of the leading-order equations up to the yet unknown (buoyancy) amplitude function $A_b = 1/N\,\hat{\Theta}^{(1)}/\hat{\theta}^{(0)}$. We obtain an evolution equation for this quantity from the first-order system (6.16) through a solvability condition: the anti-Hermitian system matrix $\boldsymbol{M}$ is singular once $\varphi$ satisfies the Hamilton–Jacobi equation for the phase field in (6.18). Multiplying the first-order equations from the left with the transpose complex conjugate of the adjoint matrices' null space vector, i.e. with

$$\left(\hat{U}^{(0)^*},\quad \hat{W}^{(0)^*},\quad A_b^*,\quad \hat{\theta}^{(0)}\hat{\Pi}^{(2)^*}\right) = \left(\mathrm{i}\frac{m}{k}\frac{\omega}{N},\quad -\mathrm{i}\frac{\omega}{N},\quad 1,\quad \mathrm{i}\frac{m}{k^2}\frac{\omega^2}{N}\right) A_b^*, \tag{6.20}$$

then eliminates the left-hand side of (6.16) and leaves us with a solvability condition for the right-hand side. The result is an evolution equation for the amplitude function $A_b(\tau, \chi, \zeta)$ which can be cast in the form of a wave-energy-conservation law,

$$\frac{\partial E'}{\partial \tau} + \nabla_{(\chi,\zeta)} \cdot \left(\boldsymbol{c}_g E'\right) = 0, \tag{6.21}$$

where, with

$$\hat{\rho}^{(0)} = \frac{\hat{\pi}^{(0)^{(1-\kappa)/\kappa}}}{\hat{\theta}^{(0)}} \tag{6.22}$$

being the non-dimensional leading-order density,

$$E' = \frac{\hat{\rho}^{(0)}}{4}\left(\left|\hat{U}^{(0)}\right|^2 + \left|\hat{W}^{(0)}\right|^2 + \frac{1}{N^2}\left|\frac{\hat{\Theta}^{(1)}}{\hat{\theta}^{(0)}}\right|^2\right) = \frac{\hat{\rho}^{(0)}}{2}\left|A_b\right|^2 \tag{6.23}$$

is the perturbation energy of the leading-order internal waves and

$$\boldsymbol{c}_g = \left(\frac{\partial \omega}{\partial k}, \frac{\partial \omega}{\partial m}\right) \tag{6.24}$$

is the group velocity of internal wavepackets. We note that the fourth equation in (6.16) is free of expressions resulting from the time derivative in the pressure equation, and thus represents a divergence constraint involving the leading- and first-order velocities. This constraint is again in the pseudo-incompressible form, so that in this regard too the pseudo-incompressible equations are consistent with the Euler equations, and we could equally well have used the former for obtaining our results.

## 7. WKB theory for large amplitudes

In § 6, we considered the evolution of small-amplitude internal waves over time scales long enough to let short-wave internal-wavepackets travel (vertical) distances comparable with the (mutually comparable) scale heights, $H$ and $H_\theta$. We found that, for two reasons, the solvability condition in the first-order theory did not involve any nonlinear terms, despite the fact that these should have appeared according to plain order-of-magnitude estimates. The two reasons for this absence of nonlinear terms in the solvability condition were (i) the solenoidality of the leading-order waves and (ii) the fact that pressure fluctuations are by one order of magnitude smaller than those of potential temperature, so that nonlinearities involving the pressure field are relegated to the next-higher order.

Here we exploit this disappearance of nonlinear terms in constructing a large-amplitude WKB theory for internal GWs near their breaking level, which will again turn out to be consistent with pseudo-incompressible theory. Notice that the regime considered corresponds with that identified in § 3.

### 7.1. *WKB expansions for the large-amplitude regime*

In line with the above findings, we consider

$$\hat{u} = \tilde{u}^{(0)}, \tag{7.1}$$

$$\hat{w} = \tilde{w}^{(0)}, \tag{7.2}$$

$$\hat{\theta} = \hat{\theta}^{(0)} + \epsilon\tilde{\theta}^{(1)}, \tag{7.3}$$

$$\hat{\pi} = \hat{\pi}^{(0)} + \epsilon^2\tilde{\pi}^{(2)}. \tag{7.4}$$

Again we make the usual WKB assumption that each wave field has local amplitude, wavenumber and frequency with only slow dependence on space and time, but now we include higher harmonics in the fast variable $\varphi/\epsilon$ to account for the possibility of nonlinear effects playing a role to first and higher orders. We also include a mean flow possibly affected by a wave mean-flow interaction. We thus assume

$$\begin{pmatrix} \tilde{u}^{(0)} \\ \tilde{w}^{(0)} \\ \tilde{\theta}^{(1)} \\ \tilde{\pi}^{(2)} \end{pmatrix} = \begin{pmatrix} \hat{U}_0^{(0)} \\ \hat{W}_0^{(0)} \\ \hat{\Theta}_0^{(1)} \\ \hat{\Pi}_0^{(2)} \end{pmatrix}(\tau,\chi,\zeta) + \mathrm{Re}\left[\begin{pmatrix} \hat{U}_1^{(0)} \\ \hat{W}_1^{(0)} \\ \hat{\Theta}_1^{(1)} \\ \hat{\Pi}_1^{(2)} \end{pmatrix}(\tau,\chi,\zeta)\exp\left(\mathrm{i}\,\frac{\varphi(\tau,\chi,\zeta)}{\epsilon}\right)\right]$$
$$+\epsilon\left[\begin{pmatrix} \hat{U}_0^{(1)} \\ \hat{W}_0^{(1)} \\ \hat{\Theta}_0^{(2)} \\ \hat{\Pi}_0^{(3)} \end{pmatrix}(\tau,\chi,\zeta) + \mathrm{Re}\sum_{\alpha=1}^{\infty}\begin{pmatrix} \hat{U}_\alpha^{(1)} \\ \hat{W}_\alpha^{(1)} \\ \hat{\Theta}_\alpha^{(2)} \\ \hat{\Pi}_\alpha^{(3)} \end{pmatrix}(\tau,\chi,\zeta)\exp\left(\mathrm{i}\alpha\,\frac{\varphi(\tau,\chi,\zeta)}{\epsilon}\right)\right] + o(\epsilon). \tag{7.5}$$

These are substituted into (3.21)–(3.24) and we again order by like powers in $\epsilon$.

### 7.2. *Leading-order analysis*

To $O(\epsilon)$ we again obtain the hydrostatic balance (3.35) from the vertical momentum equation (3.22). To next order, $O(\epsilon^2)$ for the momentum equations, $O(\epsilon)$ for the entropy equation and $O(1)$ for the Exner-pressure equation, we obtain terms multiplied by various powers of $\exp(\mathrm{i}\varphi/\epsilon)$. These are mutually orthogonal upon averaging in the fast variable $\varphi/\epsilon$ so we collect like powers.

#### 7.2.1. *First-order Fourier components*

We find from the terms proportional to $\exp(\mathrm{i}\varphi/\epsilon)$ in the Exner-pressure equation (3.24) the important solenoidality

$$\mathrm{i}k\hat{U}_1^{(0)} + \mathrm{i}mW_1^{(0)} = 0, \tag{7.6}$$

and with this altogether from (3.21) to (3.24)

$$\boldsymbol{M}(\hat{\omega},k,m)\begin{pmatrix} \hat{U}_1^{(0)} \\ \hat{W}_1^{(0)} \\ \dfrac{1}{N}\dfrac{\hat{\Theta}_1^{(1)}}{\hat{\theta}^{(0)}} \\ \hat{\theta}^{(0)}\hat{\Pi}_1^{(2)} \end{pmatrix} = 0, \tag{7.7}$$

where the same system matrix as $\boldsymbol{M}$ as in (6.15) appears, now, however, with $\omega$ replaced by the intrinsic frequency, $\hat{\omega} = \omega - k\hat{U}_0^{(0)} - m\hat{W}_0^{(0)}$. Here we also note that below $\hat{W}_0^{(0)} = 0$ is shown so the intrinsic frequency actually differs from the absolute frequency only by a Doppler term involving the horizontal flow. For the same reasons as described above, $\boldsymbol{M}$ must be singular and we obtain the GW dispersion relation

$$\hat{\omega}^2(k, m) = N^2 \frac{k^2}{k^2 + m^2}. \tag{7.8}$$

Note that neither variations in the mean-flow vertical wind nor wave-induced variations of the background stratification will enter here. From the definitions (6.9)–(6.11), one obtains the ray-tracing equations

$$\left(\frac{\partial}{\partial \tau} + \boldsymbol{c}_g \cdot \nabla_{(\chi,\zeta)}\right) k = 0, \tag{7.9}$$

$$\left(\frac{\partial}{\partial \tau} + \boldsymbol{c}_g \cdot \nabla_{(\chi,\zeta)}\right) m = -k \frac{\partial \hat{U}_0^{(0)}}{\partial \zeta}, \tag{7.10}$$

$$\left(\frac{\partial}{\partial \tau} + \boldsymbol{c}_g \cdot \nabla_{(\chi,\zeta)}\right) \omega = k \frac{\partial \hat{U}_0^{(0)}}{\partial \tau}, \tag{7.11}$$

where the group velocity

$$\boldsymbol{c}_g = \left(\hat{U}_0^{(0)} + \frac{\partial \hat{\omega}}{\partial k}, \frac{\partial \hat{\omega}}{\partial m}\right) \tag{7.12}$$

is now supplemented by the horizontal mean flow. Thus, the horizontal wavenumber is actually constant along rays defined by the local group velocity. The polarization relations are finally obtained to be (6.19), but now with $\omega$ replaced by $\hat{\omega}$.

### 7.2.2. *Mean flow*

The mean-flow contributions (zero power in $\exp(\mathrm{i}\varphi/\epsilon)$) yield the following results. From the vertical momentum equation, we obtain

$$\hat{\Theta}_0^{(1)} = 0, \tag{7.13}$$

while the entropy equation yields

$$\hat{W}_0^{(0)} = 0. \tag{7.14}$$

## 7.3. *First-order analysis*

At the next order, $O(\epsilon^3)$ for the momentum equations, $O(\epsilon^2)$ for the entropy equation and $O(\epsilon)$ for the Exner-pressure equation, we again collect terms multiplied by like powers of $\exp(\mathrm{i}\varphi/\epsilon)$. In passing we note that the terms in the Exner-pressure equation proportional to $\exp(2\mathrm{i}\varphi/\epsilon)$ yield the solenoidality

$$\mathrm{i}k\hat{U}_\alpha^{(1)} + \mathrm{i}m W_\alpha^{(1)} = 0, \quad (\alpha > 1), \tag{7.15}$$

which we will use frequently below.

7.3.1. *First-order Fourier components*

We find from the terms proportional to $\exp(\mathrm{i}\,\varphi/\epsilon)$,

$$
\boldsymbol{M}(\hat{\omega},k,m)\begin{pmatrix} \hat{U}_1^{(1)} \\ \hat{W}_1^{(1)} \\ \dfrac{1}{N}\dfrac{\hat{\Theta}_1^{(2)}}{\hat{\theta}^{(0)}} \\ \hat{\theta}^{(0)}\hat{\Pi}_1^{(3)} \end{pmatrix}
= \begin{pmatrix}
-\dfrac{\partial \hat{U}_1^{(0)}}{\partial\tau} - \hat{U}_0^{(0)}\dfrac{\partial \hat{U}_1^{(0)}}{\partial\chi} - \hat{W}_1^{(0)}\dfrac{\partial \hat{U}_0^{(0)}}{\partial\zeta} - \hat{\theta}^{(0)}\dfrac{\partial \hat{\Pi}_1^{(2)}}{\partial\chi} - \left(\mathrm{i}k\hat{U}_0^{(1)} + \mathrm{i}m\hat{W}_0^{(1)}\right)\hat{U}_1^{(0)} \\
-\dfrac{\partial \hat{W}_1^{(0)}}{\partial\tau} - \hat{U}_0^{(0)}\dfrac{\partial \hat{W}_1^{(0)}}{\partial\chi} - \hat{\theta}^{(0)}\dfrac{\partial \hat{\Pi}_1^{(2)}}{\partial\zeta} - \left(\mathrm{i}k\hat{U}_0^{(1)} + \mathrm{i}m\hat{W}_0^{(1)}\right)\hat{W}_1^{(0)} \\
-\dfrac{\partial}{\partial\tau}\left(\dfrac{\hat{\Theta}_1^{(1)}}{\hat{\theta}^{(0)}}\right) - \hat{U}_0^{(0)}\dfrac{\partial}{\partial\chi}\left(\dfrac{\hat{\Theta}_1^{(1)}}{\hat{\theta}^{(0)}}\right) - \left(\mathrm{i}k\hat{U}_0^{(1)} + \mathrm{i}m\hat{W}_0^{(1)}\right)\dfrac{\hat{\Theta}_1^{(1)}}{\hat{\theta}^{(0)}} \\
-\dfrac{\partial \hat{U}_1^{(0)}}{\partial\chi} - \dfrac{\partial \hat{W}_1^{(0)}}{\partial\zeta} - \dfrac{1-\kappa}{\kappa}\dfrac{\hat{W}_1^{(0)}}{\hat{\pi}^{(0)}}\dfrac{\partial \hat{\pi}^{(0)}}{\partial\chi}
\end{pmatrix}. \tag{7.16}
$$

Here we have already used that the leading-order mean flow is horizontally homogeneous, which is derived in (7.24) below. The further procedure is very similar to the one which led us to (6.21). Multiplying the equations above from the left with the transpose complex conjugate of the null space vector of the adjoint of $\boldsymbol{M}(\hat{\omega},k,m)$, one obtains from the real part

$$
\frac{\partial E'}{\partial\tau} + \nabla_{(\chi,\zeta)}\cdot\left(\boldsymbol{c}_g E'\right) = -\frac{1}{2}\mathrm{Re}\left(\hat{\mathrm{U}}_1^{(0)*}\,\hat{\mathrm{W}}_1^{(0)}\right)\frac{\partial \hat{\mathrm{U}}_0^{(0)}}{\partial\zeta}, \tag{7.17}
$$

where $E'$ is defined as in (6.23). The imaginary part yields a predictive equation for the large-scale and slow-time part of the wave phase $\beta = \arctan(\Im\hat{\Theta}_1^{(1)}/\mathrm{Re}\hat{\Theta}_1^{(1)})$ which, however, is not needed below. Once more using the polarization relations (6.19), with $\omega$ replaced by $\hat{\omega}$, and the ray-tracing equations (7.9)–(7.11) one finally obtains the principle of wave-action conservation (Bretherton 1966; Grimshaw 1975; Müller 1976):

$$
\frac{\partial}{\partial\tau}\left(\frac{E'}{\hat{\omega}}\right) + \nabla_{(\chi,\zeta)}\cdot\left(\boldsymbol{c}_g\frac{E'}{\hat{\omega}}\right) = 0. \tag{7.18}
$$

7.3.2. *Second-order Fourier components*

The terms proportional to $\exp(\mathrm{i}\,2\,\varphi/\epsilon)$ now include non-trivial nonlinear advection terms,

$$
\boldsymbol{M}(2\hat{\omega},2k,2m)\begin{pmatrix} \hat{U}_2^{(1)} \\ \hat{W}_2^{(1)} \\ \dfrac{1}{N}\dfrac{\hat{\Theta}_2^{(2)}}{\hat{\theta}^{(0)}} \\ \hat{\theta}^{(0)}\hat{\Pi}_2^{(3)} \end{pmatrix}
= \begin{pmatrix} -\mathrm{D}_1\hat{U}_1^{(0)} - \frac{1}{2}\mathrm{i}k\hat{\Theta}_1^{(1)}\Pi_1^{(2)} \\ -\mathrm{D}_1\hat{W}_1^{(0)} - \frac{1}{2}\mathrm{i}m\hat{\Theta}_1^{(1)}\Pi_1^{(2)} \\ -\dfrac{1}{\hat{\theta}^{(0)}}\mathrm{D}_1\hat{\Theta}_1^{(1)} \\ 0 \end{pmatrix}, \tag{7.19}
$$

where one has

$$\mathrm{D}_1 = \frac{1}{2}\left(\hat{U}_1^{(0)}\frac{\partial}{\partial\chi} + \hat{W}_1^{(0)}\frac{\partial}{\partial\zeta} + \mathrm{i}k\hat{U}_1^{(1)} + \mathrm{i}m\hat{W}_1^{(1)}\right). \tag{7.20}$$

Note that one has from (7.16)

$$\mathrm{i}k\hat{U}_1^{(1)} + \mathrm{i}m\hat{W}_1^{(1)} = -\frac{\partial\hat{U}_1^{(0)}}{\partial\chi} - \frac{\partial\hat{W}_1^{(0)}}{\partial\zeta} - \frac{1-\kappa}{\kappa}\frac{\hat{W}_1^{(0)}}{\hat{\pi}^{(0)}}\frac{\partial\hat{\pi}^{(0)}}{\partial\zeta}, \tag{7.21}$$

which can be used to eliminate $\hat{U}_1^{(1)}$ and $\hat{W}_1^{(1)}$. Then we also observe that $\boldsymbol{M}(2\hat{\omega}, 2k, 2m)$ is non-singular, because $\hat{\omega}$, $k$ and $m$ are already related by the dispersion relation in (7.8), and $\hat{\omega}(2k, 2m) \neq 2\,\hat{\omega}(k, m)$. As a consequence, the system can be solved for the unknowns $(\hat{U}_2^{(1)}, \hat{W}_2^{(1)}, 1/N\,\hat{\Theta}_2^{(2)}/\theta^{(0)}, \hat{\theta}^{(0)}\hat{\Pi}_2^{(3)})$, and no additional solvability constraint on the right-hand terms arises. Notice that the right-hand side involves the effects of nonlinear advection as well as effects of non-zero pseudo-incompressible velocity divergence as seen in (7.21).

### 7.3.3. *Higher-order Fourier components*

The higher-order terms proportional to $\exp(\mathrm{i}\,\alpha\varphi/\epsilon)$ for $\alpha > 2$ yield

$$\boldsymbol{M}(\alpha\hat{\omega}, \alpha k, \alpha m)\begin{pmatrix} \hat{U}_\alpha^{(1)} \\ \hat{W}_\alpha^{(1)} \\ \dfrac{1}{N}\dfrac{\hat{\Theta}_\alpha^{(2)}}{\hat{\theta}^{(0)}} \\ \hat{\theta}^{(0)}\hat{\Pi}_\alpha^{(3)} \end{pmatrix} = 0. \tag{7.22}$$

This homogeneous sequence of linear equations again has a non-singular system matrix, for the same reason mentioned earlier, related to the dispersion relation. As a consequence,

$$\left(\hat{U}_\alpha^{(1)}, \hat{W}_\alpha^{(1)}, \frac{1}{N}\frac{\hat{\Theta}_\alpha^{(2)}}{\hat{\theta}^{(0)}}, \hat{\theta}^{(0)}\hat{\Pi}_\alpha^{(3)}\right) = 0, \quad (\alpha > 2), \tag{7.23}$$

i.e. the higher-order terms all vanish.

### 7.3.4. *Mean flow*

Finally, the mean-flow terms of the Exner-pressure equation yield

$$\frac{\partial\hat{U}_0^{(0)}}{\partial\chi} = 0, \tag{7.24}$$

i.e. the leading-order mean flow is horizontally homogeneous. From the momentum equations and the entropy equation we obtain, again using (7.21),

$$\frac{\partial\hat{U}_0^{(0)}}{\partial\tau} + \hat{\theta}^{(0)}\frac{\partial\hat{\Pi}_0^{(2)}}{\partial\chi} = -\frac{1}{2\,\hat{\pi}^{(0)\,(1-\kappa)/\kappa}}\left\{\frac{\partial}{\partial\chi}\left(\hat{\pi}^{(0)\,(1-\kappa)/\kappa}\,|\hat{U}_1^{(0)}|^2\right) + \frac{\partial}{\partial\zeta}\left[\hat{\pi}^{(0)\,(1-\kappa)/\kappa}\,\mathrm{Re}\big(\hat{\mathrm{U}}_1^{(0)}\,\hat{\mathrm{W}}_1^{(0)^*}\big)\right]\right\} - \frac{1}{2}\mathrm{Re}\big(\mathrm{i}k\,\hat{\Theta}_1^{(1)^*}\,\Pi_1^{(2)}\big) \tag{7.25}$$

$$\hat{\theta}^{(0)}\frac{\partial\hat{\Pi}_0^{(2)}}{\partial\zeta} - \frac{\hat{\Theta}_0^{(2)}}{\hat{\theta}^{(0)}} = -\frac{1}{2\,\hat{\pi}^{(0)\,(1-\kappa)/\kappa}}\left\{\frac{\partial}{\partial\chi}\left[\hat{\pi}^{(0)\,(1-\kappa)/\kappa}\,\mathrm{Re}\big(\hat{\mathrm{U}}_1^{(0)}\,\hat{\mathrm{W}}_1^{(0)^*}\big)\right] + \frac{\partial}{\partial\zeta}\left(\hat{\pi}^{(0)\,(1-\kappa)/\kappa}\,|\hat{W}_1^{(0)}|^2\right)\right\} - \frac{1}{2}\mathrm{Re}\big(\mathrm{i}m\,\hat{\Theta}_1^{(1)^*}\,\Pi_1^{(2)}\big), \tag{7.26}$$

$$\hat{W}_0^{(1)} \frac{\partial \hat{\theta}^{(0)}}{\partial \zeta} = -\frac{1}{2\, \hat{\pi}^{(0)\,(1-\kappa)/\kappa}} \left\{ \frac{\partial}{\partial \chi} \left[ \hat{\pi}^{(0)\,(1-\kappa)/\kappa}\, \mathrm{Re}\big(\hat{\mathrm{U}}_1^{(0)}\, \hat{\Theta}_1^{(1)^*}\big) \right] + \frac{\partial}{\partial \zeta} \left[ \hat{\pi}^{(0)\,(1-\kappa)/\kappa}\, \mathrm{Re}\big(\hat{\mathrm{W}}_1^{(0)}\, \hat{\Theta}_1^{(1)^*}\big) \right] \right\}. \tag{7.27}$$

Using (6.22) and the polarization relations to be obtained from (7.7) (see also the analogous result (6.19) obtained from (6.15)), these equations can be further simplified to

$$\frac{\partial \hat{U}_0^{(0)}}{\partial \tau} + \hat{\theta}^{(0)} \frac{\partial \hat{\Pi}_0^{(2)}}{\partial \chi} = -\frac{1}{2\hat{\rho}^{(0)}} \left\{ \frac{\partial}{\partial \chi} \big(\hat{\rho}^{(0)} |\hat{U}_1^{(0)}|^2\big) + \frac{\partial}{\partial \zeta} \left[ \hat{\rho}^{(0)} \mathrm{Re}\big(\hat{\mathrm{U}}_1^{(0)}\, \hat{\mathrm{W}}_1^{(0)^*}\big) \right] \right\}, \tag{7.28}$$

$$\hat{\theta}^{(0)} \frac{\partial \hat{\Pi}_0^{(2)}}{\partial \zeta} - \frac{\hat{\Theta}_0^{(2)}}{\hat{\theta}^{(0)}} + \frac{|\hat{\Theta}_1^{(1)}|^2}{2\, \hat{\theta}^{(0)^2}} = -\frac{1}{2\hat{\rho}^{(0)}} \left\{ \frac{\partial}{\partial \chi} \left[ \hat{\rho}^{(0)} \mathrm{Re}\big(\hat{\mathrm{U}}_1^{(0)}\, \hat{\mathrm{W}}_1^{(0)^*}\big) \right] + \frac{\partial}{\partial \zeta} \big(\hat{\rho}^{(0)} |\hat{W}_1^{(0)}|^2\big) \right\}, \tag{7.29}$$

$$\hat{W}_0^{(1)} = 0. \tag{7.30}$$

The two mean-flow momentum equations, (7.28) and (7.29), demonstrate the influence of the classic divergence of the corresponding momentum fluxes. Since the zero-order vertical mean flow $\hat{W}_0^{(0)}$ vanishes, the vertical momentum equation effectively becomes a diagnostic equation for the leading-order mean-flow potential temperature $\hat{\Theta}_0^{(2)}$ induced by the wave-related momentum fluxes. Perhaps an interesting additional term is a wave-induced correction $|\hat{\Theta}_1^{(1)}|^2/2\, \hat{\theta}^{(0)^2}$ of the hydrostatic balance which already appears in the analysis of the equations in the hydrostatic limit; see (5.18). Clearly this is a term we could not derive from the anelastic equations, while it is fully contained in pseudo-incompressible dynamics! Likewise, since the first-order mean-flow potential temperature $\hat{\Theta}_0^{(1)}$ vanishes, the mean-flow entropy equation (7.27) becomes a diagnostic equation for the actually vanishing leading-order mean-flow vertical wind $\hat{W}_0^{(1)}$ induced by heating due to the actually vanishing divergence of the wave-related potential-temperature flux.

We note that at fixed wave fluxes, which are predicted from the wave-action conservation (7.18) and the GW polarization relations analogous to (6.19), (7.28)–(7.30), are four equations for as many unknowns, i.e. the horizontal mean-flow acceleration $\partial \hat{U}_0^{(0)}/\partial \tau$, the leading-order mean-flow Exner pressure $\hat{\pi}_0^{(2)}$, and $\hat{\Theta}_0^{(2)}$ and $\hat{W}_0^{(1)}$. Using (7.24), $\partial \hat{U}_0^{(0)}/\partial \tau$ can be eliminated from (7.28), yielding together with (7.29) two coupled linear equations for $\hat{\Pi}_0^{(2)}$ and $\hat{\Theta}_0^{(2)}$ which can be solved by standard means. Resubstituting $\hat{\Pi}_0^{(2)}$ into (7.28), one then obtains a predictive equation for the mean-flow horizontal wind. Finally, (7.27) gives a diagnostic relationship for the leading-order mean-flow vertical wind, which is trivially solved by (7.30) so that the mean-flow vertical wind does not only vanish to $O(1)$, but also to $O(\epsilon)$.

### 7.4. *Discussion*

This completes the analysis, which has shown that even in the large-amplitude regime favourable for internal-wave breaking, the governing equations for a WKB-type wavepacket remain those of the linear pseudo-incompressible system. The reason is simply that the equations inducing the solvability condition are identical. At the same time, we were able to explicitly write down the full first-order solutions including first-order perturbations induced by nonlinear advection, by a ‘pseudo-incompressible divergence’ and by the baroclinic torque effect. Again we stress that the divergence is indeed the one also predicted from pseudo-incompressible theory, which gives special justification for its equations in the analysis of GW dynamics in flows with arbitrary background stratification.

## 8. Summary and discussion

The scale asymptotics of the Euler equations has been examined for the case of GWs near breaking amplitude. The spatial scales taken into account are the horizontal and vertical wavelength and the potential-temperature scale height which, in the case of leading-order stratification, is comparable to the pressure scale height. Using linear theory, the wave period can be obtained from the wavelengths, thus yielding a time scale. For fixing the scales of the dynamical fields, a simple static instability criterion has been used to set the magnitude of the potential-temperature fluctuations. The polarization relations from linear theory have been used to determine therefrom the magnitude of the wind fluctuations. Non-dimensionalizing the equations in this way, it is shown that a single non-dimensional parameter remains, which is given by the ratio between wavelength and the potential-temperature scale height. An expansion in this small parameter yields an equation hierarchy which turns out to be the same, to leading orders, as to be obtained from the pseudo-incompressible equations. In this fashion, consistency between the Euler equations and the pseudo-incompressible equations is shown, at least for conditions favourable to GW breaking. We emphasize that this result is independent of the strength of the background stratification.

On the contrary, the anelastic equations can be shown to be consistent only for weak stratification which, however, does not prevail at high altitudes where internal waves tend to break. We take this as an indication that a safe option for sound-proof studies of GW dynamics beyond simplified qualitative linear theories is offered by the pseudo-incompressible equations, while caution might be in place with regard to the anelastic models. We note, however, that, consistent with the present findings, several studies have found the anelastic theory to perform as well as the pseudo-incompressible model under conditions of sufficiently weak stratification (Nance 1994; Nance & Durran 1994; Bannon 2001; Davies *et al.* 2003; Klein 2009; Klein *et al.* 2010), and still quite well even under more general conditions (Prusa, Smolarkiewicz & Wyszogrodzki 2008). Nonetheless, weaknesses have been identified, at least in the non-hydrostatic case, which the pseudo-incompressible theory does not exhibit.

Using the multiple-scale asymptotics technique of WKB expansions, we have developed reduced dynamical equations for small-scale internal GW packets as they travel large vertical distances comparable to the pressure scale height. Consistent leading- and first-order solutions have been constructed that are valid for amplitudes in the wave-breaking regime. The leading-order solutions are governed by linear pseudo-incompressible dynamics. At first order, we obtain explicit expressions for the influence of nonlinear advection. The analysis was facilitated by the fact that pressure fluctuations even in this large-amplitude regime turn out to be negligible in the Exner-pressure equation. As a consequence, the velocity field remains divergence-free at leading order and is thus solenoidal. An interesting side result is that in the WKB theory only variations of the horizontal mean flow play an important role. Neither the vertical wind nor wave-induced variations of the stratification have a leading-order impact on the wave properties. It might also be interesting that the mean flow is not only influenced by the divergence of the wave-related momentum flux, but also by a wave-induced correction of hydrostatic balance, which appears neither in Boussinesq nor in anelastic theory.

A recent related study is the one by Shaw & Shepherd (2008, 2009). They discuss closure schemes for the net nonlinear planetary scale fluxes induced by mesoscale flow fluctuations through wave-action and pseudo-momentum flux terms. While their analysis is based on the general Hamiltonian formulation of the anelastic and compressible dynamical equations, it does not provide explicit WKB-type solutions and a comparison of the anelastic and pseudo-incompressible models. An interesting difference is that they consider mean-flow scales which seem to directly yield a simple hydrostatic equilibrium from the vertical momentum balance which is not influenced by wave fluxes. The development of a unifying framework for this study and that by Shaw & Shepherd (2008, 2009) is an interesting task for the future. Then rotation could no longer be neglected, and the interplay between inertia and gravitation would have to be considered. Similarly, one might be interested in cases with an interaction between GWs and acoustic waves on the one hand or Rossby waves on the other. Corresponding analyses and comparison with direct integrations of the compressible and pseudo-incompressible model equations will also be left for future study.

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